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Reliability of the Weibull analysis of the strength of construction materials

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Abstract The breaking force during transversal loading of fibre-cement corrugated roofing sheets was measured on several test samples from a serial production. The results were statistically analyzed assuming the 2-parameter Weibull statistics. In addition, Monte Carlo statistical simulations were made by using a computerised built-in random-number generator. While smaller sample data groups, mostly containing up to 50 samples, were studied in the literature, we extended their size up to 400 samples. We showed that some trends in the evaluation of statistical parameters which hold for smaller data groups, apply well to larger data groups. In particular, we confirmed that the statistical distribution of the Weibull parameters obtained from repeated Monte Carlo simulations is log-normal. Furthermore, we considered the influence of the measurement uncertainty on the statistical parameters.

Introduction

In recent years there have been numerous successful investigations in civil engineering relating to the development of fibre-cement composites using various organic or synthetic fibres [1–6]. However, during the development stage the new composite materials and their products have to be tested rigorously using classical mechanical tests, such as the 3-point bending test [7, 8]. In addition, in order to control the production quality mechanical tests should be regularly performed on samples from serial production.

The strengths measured in typical mechanical tests for brittle materials, e.g., ceramics, cement and concrete, result in a Weibull statistical distribution [9–11]. This distribution has been widely used not only in civil engineering [12–16], but also in many other fields, with many examples from the literature given in [11]. Usually, a simple 2-parameter Weibull statistics is used, although in many cases the application of 3-parameter Weibull statistics or Weibull statistics corresponding to two or more different fracture modes gives better results [11, 15, 17, 18].

The validity of the Weibull parameters' estimations has been investigated experimentally and theoretically [17–34]. For instance, Orlovskaja et al. divided the 137 strength values obtained in the bending tests of a recrystallised silicon carbide ceramic into 10^5 random smaller subsets and studied the corresponding statistics on the different values of the Weibull parameters from several subsets [23]. They found a good agreement in the variation of the scale parameter with previous theoretical predictions [22] but the agreement for the Weibull modulus was worse and this was attributed to the deviation of the strength statistics from ideal Weibull statistics.

Several authors used Monte Carlo simulations with prescribed (input) Weibull parameters to obtain sample data (instead of real experimental data) with the following aims: (a) to compare the results for different procedures of estimating Weibull parameters [32], (b) to optimise the simple probability estimator functions in the linear regression method in order to minimise the biasing and/or estimation uncertainty of the Weibull parameters [25–33], (c) to analyze the type of the distribution functions for the large sets of estimated Weibull parameters [19, 21, 24, 26].

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One of the important results of such studies is, for instance, that the distribution functions of the estimated Weibull parameters from several Monte Carlo repetitions obey the log-normal distribution which is considerably asymmetrical for small sizes of sample data groups [21, 24, 26]. It has also been pointed out that the statistical distribution of the ratio m_{est}/m (where m_{est} is the estimated Weibull modulus for a single data group in one Monte Carlo simulation and m is the input Weibull modulus) in the series of repeated Monte Carlo simulations is independent of the choice of particular values of input Weibull parameters [19, 21], except for the moments method [32]. Most Monte Carlo studies have been limited to small sizes of data groups, up to 50, since most experimental measurements are limited to such data-group sizes.

In this article the results of measurements of the transverse breaking force on corrugated roofing sheets, i.e., the samples from the serial production, are presented. The experimental data are fitted to the 2-parameter Weibull probability distribution function. The first aim of this paper is to show that the 2-parameter Weibull statistics can be used successfully in this case. Furthermore, we confirm the validity of the theoretical assumption that the estimated positive Weibull parameters for several sample groups of finite size obey the log-normal distribution. For this task a computerised random-number generator is used. The validity of the estimated confidence bounds on Weibull parameters (which depend on the size of the testing group) is checked and some results are compared to those in the literature. We extend the study of small data groups to the sizes of sample data groups up to 400. Finally, the significance of the measurement uncertainty to the statistical results is evaluated.

Experiment

The material composition and the manufacturing process for the corrugated roofing sheets based on the Hatschek procedure are described elsewhere [1, 35, 36]. In short, the sheets are made of a fibre-cement composite with short-cut polyvinyl alcohol (PVA) fibres in our case. The sheet dimensions are: width $W = 920$ mm, length $L = 1250$ mm, corrugation pitch $P = 177$ mm, corrugation height $H = 51$ mm (Fig. 1). The typical thickness, T, of the sheets is between 5.95 mm and 6.20 mm. This is the product dimension which is the most difficult to control and contributes to the scatter of the experimental force data (in addition to material's inherent strength and the space distribution of flaws in the material). Before the mechanical measurements the test sheets were soaked in water for 24 h, in order to simulate the influence of bad weather conditions on their mechanical properties.

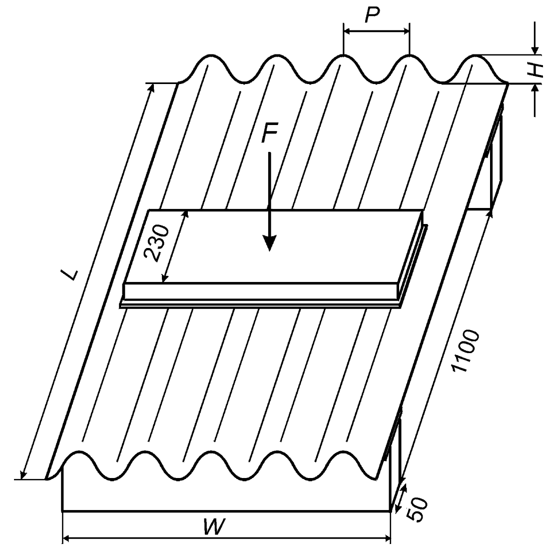

Fig. 1 The geometry of transverse loading according to the European standard EN 494. Lengths are given in milimeters

For the determination of the mechanical properties of the sheets the methods prescribed by the European standard EN 494 were used [8]. In this paper we focus on the measurement of the breaking force, F, during transversal loading with respect to the sheet corrugations, which is essentially a 3-point bending test: the load is slowly increased until the sample sheet breaks. The prescribed distances are shown in Fig. 1. The strips of soft material, for instance felt, are recommended to be inserted between the sheet and the loading and supporting bars. The bars must be longer than the sheet width and must be aligned perpendicularly to the sheet corrugations as well as possible. The load rate should be such that the rupture of the specimen occurs between 10 s and 45 s after beginning of loading. For these measurements a BP-10 laboratory press-machine (Walter + Bai AG, Switzerland), with the measuring scales 2 kN and 10 kN, and equipped with the corresponding software, was used. The measurement uncertainty of the breaking force was estimated to 30 N, i.e., relative uncertainty is about 0.5% for a typical breaking force of 6 kN. But the actual uncertainty may be higher if additional effects, such as the sample misalignment, are considered.

Statistical evaluation of the data

Our statistical variable is the transversal breaking force (shortly called the force), F, with the probability density function $p(F)$. The cumulative probability function, also called the unreliability function, is defined as: $P(F) = \int_0^F p(F')dF'$, which means the probability of finding the value for the measured force to be less than F [11].

In the case of 2-parameter Weibull statistics, the functions $p(F)$ and $P(F)$ are equal to:

$$p(F) = \frac{m}{F_0}\left(\frac{F}{F_0}\right)^{m-1}\exp\left(-\left(\frac{F}{F_0}\right)^m\right) \tag{1a}$$

$$P(F) = 1 - \exp\left(-\left(\frac{F}{F_0}\right)^m\right), \tag{1b}$$

with the pair of Weibull parameters m and F_0. The dimensionless Weibull modulus, m, and the scale parameter, F_0, determine the slope and position of the straight line in the linear diagram $y(x)$, which represents the $P(F)$ dependence in the rescaled variables, $x = \ln F$ and $y = \ln(\ln(1/(1 - P)))$.

After measuring N values of the force, F_i, $i = 1$ to N, the data are fitted to the unreliability function (1b), in order to estimate the Weibull parameters m and F_0. Here, a rough sketch of the procedure is given; the reader can find more details elsewhere [11, 37–39]. The N values F_i are first sorted in increasing order. Then the value P_i corresponding to the unreliability function is attributed to the i-th value F_i (P_i being independent of the value x_i) by solving numerically the equation:

$$\sum_{k=i}^{N}\binom{N}{k}P_i^k(1-P_i)^{N-k} = 0.5, \tag{2a}$$

in accordance with the binomial distribution [38]. Alternatively, a much simpler equation is often used, which leads to similar calculated values for P_i:

$$P_i = \frac{i-0.3}{N+0.4} \tag{2b}$$

Other simple formulas are also used in the literature, instead of Eq. 2b [11, 15, 25–33, 40]. Finally, the ordered data pairs (x_i, y_i), with $x_i = \ln F_i$ and $y_i = \ln(\ln(1/(1 - P_i)))$, are fitted to the straight line. The best-fit criterion is the minimum sum of either squared horizontal distances (X-regression), or squared vertical distances (Y-regression) between the line and the data points.

Knowing the Weibull parameters one can calculate various statistical parameters, for instance, the theoretical mean value $<F>_\infty$ of the force and its standard deviation $\delta_{F\infty}$, corresponding to the limit $N \to \infty$, and compare them to the actual values, $<F>_N$ and σ_{FN}, for a finite sample group of size N. The relevant formulae are:

$$<F>_\infty = F_0 \cdot \Gamma\left(1+\frac{1}{m}\right) \tag{3a}$$

$$\sigma_{F\infty} = F_0 \cdot \sqrt{\Gamma\left(1+\frac{2}{m}\right) - \Gamma^2\left(1+\frac{1}{m}\right)}, \tag{3b}$$

where $\Gamma = \int_0^\infty t^{x-1}e^{-t}dt$ is the gamma function, and:

$$<F>_N = \frac{1}{N}\sum_{i=1}^{N}F_i \tag{4a}$$

$$\sigma_{FN} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(F_i - <F>)^2}. \tag{4b}$$

The estimation of the confidence intervals of the calculated Weibull parameters is usually done with the help of the Fisher matrix of the system [39] and some details of this procedure are given in the Appendix. One can then predict with some confidence level CL (probability), that the actual value of the parameter in question lies within some interval (confidence bounds). A confidence level CL = 90% is often used. In the calculation of the confidence bounds of m and F_0 from the Fisher matrix corresponding to the set of N data points it is assumed that the distribution of the estimated values of the parameters is log-normal. This means that if there are several groups of size N of independent experimental data, and for each group different values of m and F_0 are obtained, the resulting distribution of their natural logarithms is Gaussian:

$$p(\ln x) = \frac{1}{w\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{\ln x - a}{w}\right)^2\right), \tag{5}$$

where x stands either for m or for F_0; let us take it to be m. Thus the most probable value of m is estimated as $m = \exp(a)$. The lower bound, m_{LB}, and the upper bound, m_{UB}, of the confidence interval corresponding to a given confidence level, CL, are given as: $m_{UB}, m_{LB} = m \cdot \exp(\pm w \cdot erf^{-1}(\mathrm{CL}))$, where the plus and minus sign refer to m_{UB} and m_{LB}, respectively, and the error function is defined as:

$$erf(x) = \sqrt{\frac{2}{\pi}}\int_0^x \exp(-t^2/2)dt.$$

The same procedure is used for the estimation of the scale parameter F_0 and its confidence bounds, $F_{0,LB}$ and $F_{0,UB}$. It should be noted that the confidence bounds can be calculated for other statistical parameters as well, e.g., for the mean value and standard deviation of the force, and even for the $P(F)$ dependence: more specifically, the fraction of broken test samples for a given load [17].

Model

For comparison we use three different approaches to evaluate or model the experimental data as described below.

Using experimental data and the Fisher matrix

Different groups of different sizes N out of available experimental force data (460 values altogether) are inserted into ReliaSoft's commercial computer program package Reliasoft-Weibull++ for the computation and visualisation of the statistical distribution [11]. This program uses the formula (2a) to obtain the N values P_i, and then the linear function is searched that best fits the N points with rescaled variables (x_i, y_i). Both regression types, X and Y, give slightly different values of estimated parameters. Here we present the results for the Y-regression. The Fisher matrix is used to obtain the "half-widths" w_m and w_{F0} of the Gaussians (5) for both parameters and finally the confidence bounds for m and F_0 are calculated.

Mixing experimental data using the computer random generator

The procedure of dividing the experimental data into smaller random subsets (subgroups) was used before, for instance in Ref. [23]. We have written our own computer program for this task, where both formulae, (2a) and (2b), to get the probabilities P_i, and both types of regression, X and Y, are used. For a chosen N, say $N = 50$, we repeat several (typically $N_{\mathrm{rep}} = 5 \cdot 10^4$) times the following procedure. We use the computerised built-in random-number generator (giving homogeneously distributed numbers from 0 to 1) to "shuffle" the 460 experimental force values and choose the N values out of them. Each time we obtain a different pair of values for m and F_0, as described above, thus we have altogether N_{rep} such pairs. We finally fit the distribution of their logarithms to the Gaussian function (5), to obtain the most probable values of the Weibull parameters and the corresponding confidence intervals.

Generating data by random generator (Monte Carlo simulation)

This procedure is similar to that described in (Mixing experimental data using the computer random generator) except that instead of experimental data we generate the Monte Carlo data for the forces, using the prescribed input Weibull parameters m_{inp} and $F_{0,\mathrm{inp}}$, similarly as was done, for instance, in Refs. [25–33]. A "random" force is calculated from the given random number r, using Eq. 1b, by:

$$F = P^{-1}(r) = F_{0,\mathrm{inp}} \cdot \sqrt[m_{\mathrm{inp}}]{\ln\left(\frac{1}{1-r}\right)}. \tag{6a}$$

Taking N_{rep} statistical repetitions and the group size N we get $N_{\mathrm{rep}} \cdot N$ independent forces altogether and N_{rep} different pairs of m and F_0. Finally, we obtain the most probable values of m and F_0 and the corresponding confidence intervals in the same manner as described in (Mixing experimental data using the computer random generator). Until now the measurement of the transversal breaking forces was assumed to be precise. To our knowledge, the measurement errors have always been neglected in the literature. We can also simulate the measurement uncertainty by taking the Gaussian distribution of measuring errors:

$$p(\Delta F) = \frac{1}{w_F\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{\Delta F}{w_F}\right)^2\right),$$

where ΔF means the error of the individual measurement and w_F is the measurement uncertainty. We include the measurement errors in the Monte Carlo simulation by:

$$F = F_{0,\mathrm{inp}} \cdot \sqrt[m_{\mathrm{inp}}]{\ln\left(\frac{1}{1-r_1}\right)} \pm w_F \cdot erf^{-1}(|2r_2 - 1|), \tag{6b}$$

where r_1 and r_2 are two independent random numbers. In the second term the negative or positive sign is chosen, whether $r_2 < 0.5$ or $r_2 > 0.5$, respectively.

Results and discussion

Several force values were collected in the years 2003–2005 and in the present work 460 values for non-coloured sheets are selected out of them. For instance, since it can proved that the production in the three summer months, from June to August, results in a lower mean force compared to the rest of the year only the data from the other 9 months are used to get the above mentioned 460 values.

We compare the results for several testing samples (either on experimental data or in Monte Carlo simulation) and for different ways of fitting the data to the Weibull statistics. We find a negligible difference in the Weibull parameters when Eqs. 2a or b are used for probabilities P_i, respectively. On the other hand, there is a small, but evident difference between the results from X and Y regression in the fitting procedure: on average X regression gives slightly larger Weibull modulus and we are not able to explain this fact. All results in this paper are presented for the simple formula (2b) (except for the results from the program Reliasoft-Weibull++, which uses formula (2a)) and Y regression.

Non-mixed experimental data and the Fisher matrix

First, all the 460 force values are taken and the following parameters are obtained: $m = 12.40$, $F_0 = 6293$ N, with the correlation coefficient $\rho = 0.9835$; since ρ is nearly 1, this means a very good fitting of experimental data to the Weibull statistics. A consistent result is obtained for the mean force: $\langle F\rangle_\infty = \langle F\rangle_N = 6038$ N. However, the standard deviations, calculated by (3b) and (4b) differ: $\sigma_{F\infty} = 592$ N and $\sigma_{FN} = 602$ N, and the possible reasons for this disagreement will be examined later on. Using the Fisher matrix the 50% and 90% CL bounds for m and F_0 are calculated. For instance, the 90% CL interval for m is (11.77, 13.07), i.e., its width is about 10% the estimated value for m. On the other hand, the 90% CL interval for F_0 is (6250 N, 6336 N); its width is only about 1% of the estimated value for F_0.

In the next step the dependence of confidence intervals on the size N of the testing group is investigated. The N sample values, e.g., $N = 50$ data, are chosen out of total 460 data in the same order as obtained in the measurements, from the beginning of year 2003. The width of the confidence intervals decreases with increasing N, as should be the case. Figure 2 shows the N-dependence of the estimated Weibull parameters and their 50% and 90% CL bounds. Kinks in the curves indicate possible seasonal fluctuations in the quality of the products which may arise because of the limited control of some production parameters, such as the quality of raw materials, temperature, moisture, etc. Nevertheless, a slight increase of m and decrease of F_0 with N can be noticed.

Mixing experimental data

In the next step we "shuffle" well the experimental data by random generator. The distribution of the values of both Weibull parameters fit the log-normal function excellently in all cases. Even in the case $N = 400$ where the "half-width" of the distribution function is narrow it was evident that fitting to log-normal function works better than fitting to Gaussian. In Fig. 3a,c the estimated parameters for non-mixed and mixed experimental data are compared; the data points for parameters according to mixed experimental data are given as the most probable values, see Eq. 5. The variations of the estimated parameters with N for mixed data are much smaller in comparison to non-mixed data and this can be attributed to the use of the most probable values of the parameters in the case of mixed data. Nevertheless, a trend of slightly increasing m and decreasing F_0 remains. Figure 3b,d compare the widths of the corresponding confidence intervals: $\Delta m = m_{UB} - m_{LB}$, $\Delta F_0 = F_{0,UB} - F_{0,LB}$. The values for non-mixed and mixed data are similar except for large N. Perhaps, it is not completely correct to obtain N data from finite amount of the same data in each of N_{rep} repetitions instead of taking independent data each time, especially when taking randomly, from instance, 400 data from only 460 different values. Nevertheless, this procedure at least reveals the log-normal distribution of the Weibull parameters and shows qualitatively consistent N-dependence of the estimated Weibull parameters and their confidence intervals. This is most probably due to the fact that we can still get an enormous number of combinations (compared to N_{rep}) in selecting, for instance, 400 values out of 460.

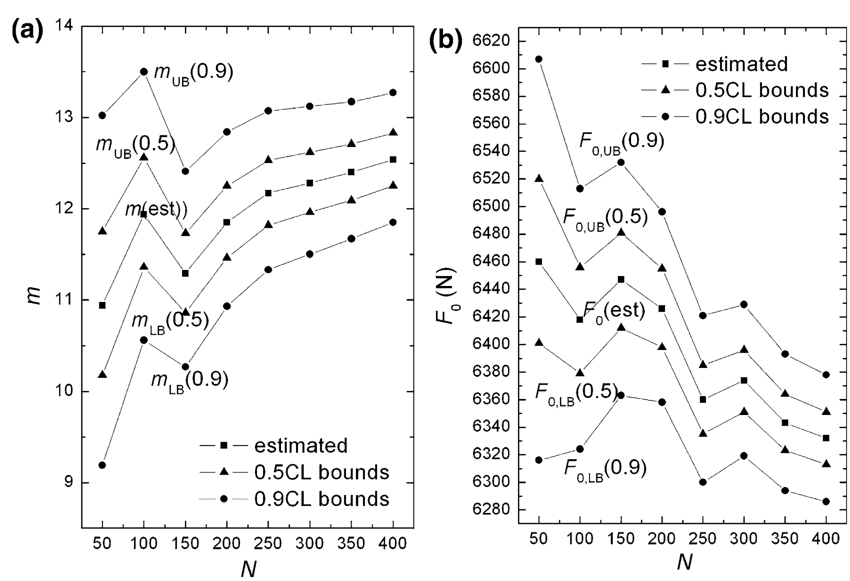

Fig. 2 Dependence of the estimated Weibull parameters from the non-mixed experimental data and the corresponding 50% and 90% CL bounds on the testing group size N: (**a**) Weibull modulus, (**b**) scale parameter

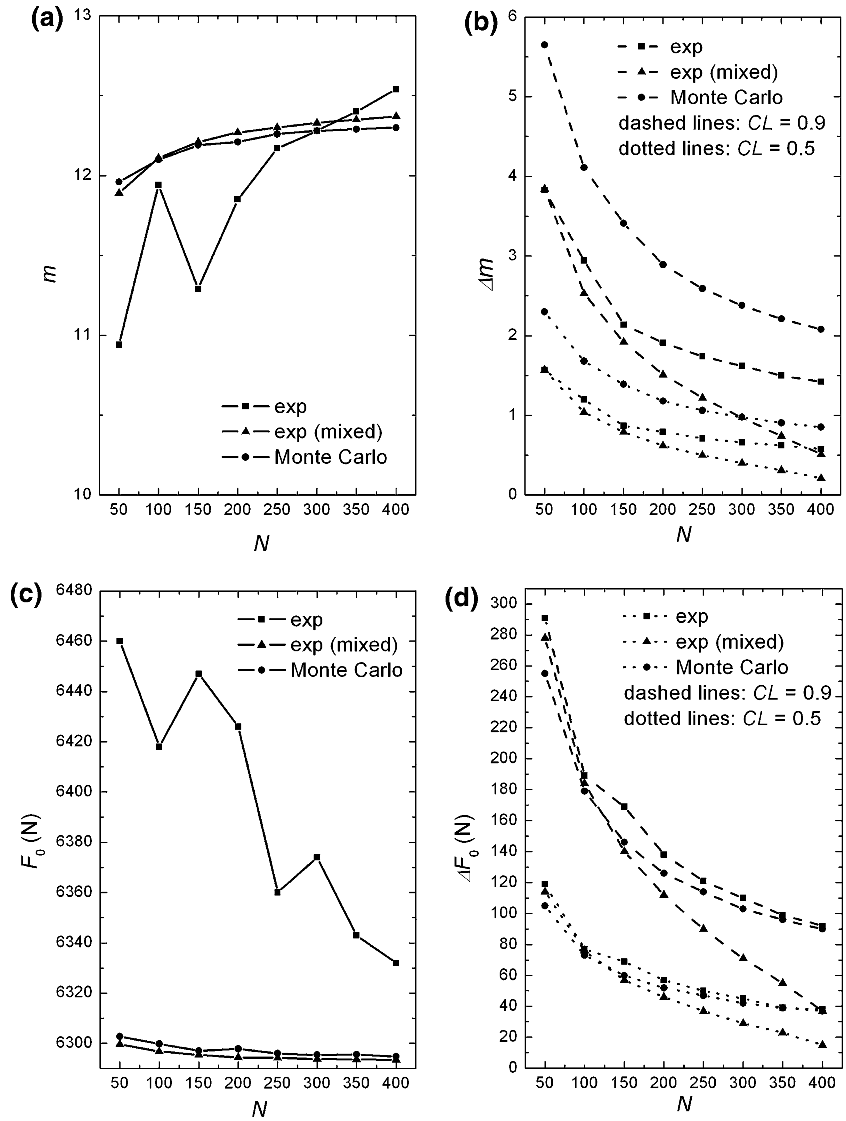

Fig. 3 Comparison of the estimated Weibull parameters and the widths of the corresponding 50% and 90% CL intervals for the non-mixed experimental data, mixed (shuffled) experimental data and Monte Carlo data for different N: (**a**) estimated m, (**b**) width of the confidence intervals Δm, (**c**) estimated F_0, (**d**) width of the confidence intervals ΔF_0. The data points for parameters according to mixed experimental and Monte Carlo data in Figs. (**a**) and (**c**) are given as the most probable values, see Eq. 5

Monte Carlo simulations

Next, we make a Monte Carlo simulation of the force data, first with no measurement error, Eq. 6a. The input parameters $m_{\mathrm{inp}} = 12.40$ and $F_{0,\mathrm{inp}} = 6293$ N are used which are believed to be near the true values since these are the values estimated for all $N = 460$ data points. Usually we take $N_{\mathrm{rep}} = 5 \cdot 10^4$, however to confirm the corresponding statistical results we make from time to time a test with even 10-times larger N_{rep}. The estimated Weibull parameters and the widths of the corresponding confidence intervals are compared to those for experimental data in Fig. 3. The estimated values of m and F_0 for Monte Carlo simulation and mixed experimental data differ very little. Such a good agreement suggests that the method of mixing experimental data described in (Mixing experimental data using the computer random generator) is a reliable strategy to obtain Weibull parameters. It is also clearly seen that using the formula (2b) slightly underestimates the true value of the Weibull modulus m, as was already claimed by Wu et al. [32, 33] and other authors. Using Monte Carlo simulations, they showed that in the case of small N the simple formulas like (2b) give slightly biased values of Weibull moduli, either too high or too low values. For $N = 50$ we get $m = 11.96$ instead of the "true" value 12.40, that is a 3.5% difference, while F_0 is overestimated by 0.2%; these estimation errors decrease slowly with N, as shown in Fig. 4. Consequently, the calculated mean force $\langle F \rangle_\infty$ and its standard deviation $\sigma_{F\infty}$ from Eqs. 3, deviate from the values, corresponding to the input parameters m_{inp} and $F_{0,\mathrm{inp}}$, as shown in Fig. 4. Figure 4 also shows the agreement with the Monte Carlo study of Wu et al. where the accuracy of the estimated scale parameter was

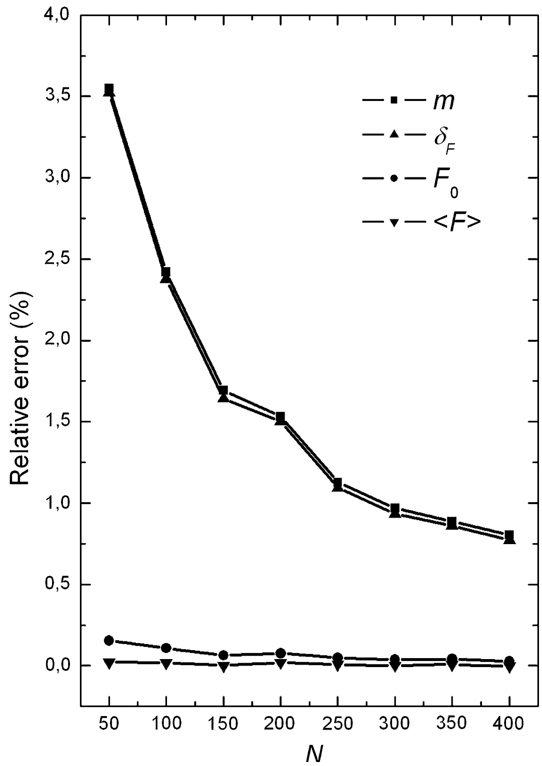

Fig. 4 The N-dependence of the error in the estimated statistical parameters from Monte Carlo simulations due to biasing the parameters by using Eq. 2b: m, F_0, $<F>_\infty$ and $\sigma_{F\infty}$. Input parameters: $m_{\text{inp}} = 12.40$, $F_{0,\text{inp}} = 6293$

shown to be about an order of magnitude higher compared to Weibull modulus [32].

Peterlik, Orlovskaja et al. found that the average modulus is overestimated while the scale parameter is underestimated in the statistical procedure of dividing the fundamental data set into smaller subsets [22, 23], contrary to our results and those of Wu et al. [32] (and other authors). The most probable reason for this disagreement is their use of maximum-likehood method to estimate the Weibull parameters instead of the linear regression method described above. It is well known that different methods give different results [11, 25–33] and our calculations confirm this fact. It should be also mentioned that the direct arithmetic averaging of the Weibull parameters over statistical repetitions, as described in Ref. [23], is not completely correct; their natural logarithms should be averaged instead since their distribution is log-normal. Direct arithmetic averaging brings a factor $\exp(w^2/2)$ to the "true" value of the Weibull parameter, see Eq. 5, and this can be also verified by Monte Carlo simulation. But this factor is only slightly greater than 1 for both parameters and cannot explain the differences in the biaxing of the Weibull parameters in Ref. [23] and our work.

As regards the confidence bounds of the parameters there is a significant discrepancy between the results from Monte Carlo simulations on one hand, and from experimental data (mixed or not) on the other hand. The Monte Carlo simulations give larger confidence intervals (Fig. 3 c,d), thus the estimations of the confidence bounds from a limited number of data must be taken with some reservation. The comparison of the confidence interval widths from experimental and Monte Carlo data, such as those in Fig. 3c,d could serve to determine the correction factors for the confidence bounds obtained with the use of the Fisher matrix of the limited amount of experimental data. Furthermore, the standard deviations of the Weibull parameters which can be calculated from the widths of the corresponding log-normal distributions (5) do not fit the formulae suggested in Refs. [22, 23] and, by our opinion, the reason for this disagreement is the same as described above for the disagreement in the biasing of the parameters. Moreover, we compared our "half-widths" w of the distribution function (5) for the Weibull modulus with the simple fitting formula in Ref. [24] where the author used the linear regression method to obtain Weibull paramers, as we did. Although the author studied the maximum size of $N = 50$ only, his fitting formula agrees reasonably well with our results even up to $N = 400$, where the deviation is only 5%.

In all Monte Carlo simulations we calculate the mean force and its standard deviation in both ways, using Eqs. 3 and 4, to check their consistency. $<F>_\infty$ and $\sigma_{F\infty}$ are calculated only at the end of the Monte Carlo simulation, by using m and F_0 as the most probable parameters in the log-normal distribution. $<F>_N$ is obtained as the average over all $N_{\text{rep}} \cdot N$ forces in the simulation, while σ_{FN} is calculated for each repetition (group size N) and then averaged over all N_{rep} values. While $<F>_\infty$ and $<F>_N$ agree very well, $\sigma_{FN} < \sigma_{F\infty}$ always holds. For $N = 50$ the relative difference between σ_{FN} and $\sigma_{F\infty}$ is about 4.3% and decreases with N, being only 1% for $N = 400$.

Monte Carlo simulations considering measurement uncertainty

In order to investigate the influence of the measuring error on the evaluation of statistical parameters we repeat Monte Carlo simulations, now using Eq. 6b with the estimated measurement uncertainty $w_F = 30$ N, taking again $m_{\text{inp}} = 12.40$ and $F_{0,\text{inp}} = 6293$ N. The differences of all statistical parameters in comparison to $w_F = 0$ are found to be negligible for all investigated N up to 800. Even for an exaggerated measurement uncertainty $w_F = 100$ N the parameters change a little. Here, we give the calculated parameters for comparison in the case $N = 460$:

- $w_F = 0$: $m = 12.32$ (90% CL interval from11.39 to 13.32), $F_0 = 6295$ N (90% CL interval from 6253 N to 6337 N), $<F>_\infty = <F>_N = 6038$ N, $\sigma_{F\infty} = 596$ N, $\sigma_{FN} = 591$ N
- $w_F = 100$ N: $m = 12.15$ (90% CL interval from11.25 to 13.13), $F_0 = 6298$ N (90% CL interval from 6255 N to 6340 N), $<F>_\infty = <F>_N = 6038$ N, $\sigma_{F\infty} = 604$ N, $\sigma_{FN} = 599$ N

Only when w_F becomes comparable to $\sigma_{F\infty}$, does the measurement uncertainty influence the statistical results significantly. The (virtual) decrease of m with increasing w_F has been expected since the measurement uncertainty widens the force distribution function, similarly as lower m would do. Here we must emphasise that although the measurement uncertainty has been incorporated in the simulation by (6b) the inverse procedure of getting the Weibull parameters from the Monte Carlo force data is still based on the function (1b) for exact measurement.

There remains a question of the correct interpretation of the 1,7% discrepancy between σ_{FN} and $\sigma_{F\infty}$ for $N = 460$ experimental data as mentioned at the beginning of this section: $\sigma_{FN} > \sigma_{F\infty}$, in contrast to predictions from Monte Carlo simulations, which at least on average give $\sigma_{FN} < \sigma_{F\infty}$, even when the measurement error is included. To explore this question further we repeat Monte Carlo simulations for $N = 460$, but now we compare σ_{FN} and $\sigma_{F\infty}$ in each of the N_{rep} repetitions separately. Although $\sigma_{FN} < \sigma_{F\infty}$ holds on average, the opposite is found to be the case in 7.8% of Monte Carlo repetitions for $w_F = 0$ (no measurement uncertainty), in 9.6% repetitions for $w_F = 30$ N, and in 11.4% for $w_F = 100$ N. This gives a relatively small (but not at all negligible!) probability of finding $\sigma_{FN} > \sigma_{F\infty}$ in a particular case of experimental data. We conclude that $\sigma_{FN} > \sigma_{F\infty}$ in our case could be explained by uncertainties of the statistical parameter estimations in the procedures described above, and that the measurement errors have a less significant influence on the result.

Application of the 3-parameter Weibull distribution

Of course, another, simpler explanation can be found for the $\sigma_{FN} > \sigma_{F\infty}$ discrepancy. Perhaps the experimental data may be better described by another statistical distribution. We have also used the 3-parameter Weibull distribution where the third parameter F_{sh} means the shift of the breaking force, $F \rightarrow F - F_{\mathrm{sh}}$, in Eq. 1. Physically this means that theoretically the smallest breaking force is F_{sh}. Fitting the 460 experimental data to the 3-parameter Weibull statistics in the Reliasoft's program results in the following parameters: $m = 4.16$, $F_0 = 2456$ and $F_{\mathrm{sh}} = 3808$. The correlation coefficient is now $\rho = 0.9945$, higher than for 2-parametric Weibull distribution since 3 parameters give higher flexibility to fit the experimental data than 2 parameters. The new parameters give $<F>_\infty = 6039$ N and a consistent relation $\sigma_{F\infty} = 604$ N $> \sigma_{FN}$. These results do not mean that the description of the experimental data with the 2-parameter Weibull statistics is essentially wrong since there is no obvious physical reason that the 3-parameter Weibull statistics is valid.

Conclusions

A study of the statistical distribution of the measured values of the breaking force in the transversal 3-point bending test of corrugated roofing sheets has been made. The measurements can be fitted well to the 2-parameter Weibull statistical distribution since the correlation coefficient is close to the value 1 in all cases. The comparable Monte Carlo simulations were also made, to reveal that the distribution of the Weibull parameters from independent groups of data (of equal size) is log-normal. However, the actual sizes of the confidence intervals of the estimated parameters seem to be much larger than those estimated from the finite amount of data by using the Fisher matrix. Another applied procedure, somewhere between that of using the Fisher matrix on the group of experimental data and the Monte Carlo approach, is repeatedly mixing and extracting the experimental data and obtaining the confidence bounds of the parameters directly from their statistical distributions. The latter procedure, although not rigorously justified, indicates that the problem of the correct determination of the confidence intervals lies in the limited number of experimental data, but not in use of the Fisher matrix.

The measurement uncertainty is “screened” within the width of the distribution function of the measured quantity. In our case it has a negligible effect on the evaluated statistical parameters, and it becomes significant when the order of magnitude of the measurement errors approaches the inherent standard deviation of the quantity. Of course, in that regime the use of Eq. 1b, which neglects the measurement uncertainty, is not applicable. However, it is in principle possible to decouple the Weibull distribution from the measurement uncertainty directly from given experimental data, and this is the subject of our future work.

Finally, it should be emphasised that the problems described above, together with their possible solutions using time-effective Monte Carlo simulations (which can be run on a personal computer), are related not only to the specific problem of the Weibull statistics of breaking forces, but have a much wider dimension. For instance, as long as any free parameter of any statistical distribution (not only the Weibull statistics) is strictly positive, it is to be expected that its statistical distribution will be log-normal.

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Appendix: The Fisher matrix and the parameter confidence bounds

First the logarithmic likelihood function Λ is defined which can also be used for the estimation of the Weibull parameters according to the maximum likelihood method:

$$\Lambda(m, F_0) = \sum_{i=1}^{N} \ln p(F_i, m, F_0),$$

where N is the number of measurements, p is the distribution function (1a), F_i is the i-th breaking force, and the parameters m and F_0 can still be varied. The Fisher information matrix F is defined by the second derivatives of the function Λ with respect to Weibull parameters:

$$\underline{\underline{F}} = \begin{bmatrix} \frac{\partial^2\Lambda}{\partial m^2} & \frac{\partial^2\Lambda}{\partial m \partial F_0} \\ \frac{\partial^2\Lambda}{\partial m \partial F_0} & \frac{\partial^2\Lambda}{\partial F_0^2} \end{bmatrix}$$

When the estimated Weibull parameters (obtained, for instance, by the linear regression method) are inserted into the Fisher matrix which is afterwards inverted the covariance matrix C is obtained which consists of variances and covariance of the Weibull parameters:

$$\underline{\underline{C}} = \begin{bmatrix} Var(m) & Cov(m, F_0) \\ Cov(m, F_0) & Var(F_0) \end{bmatrix} = \underline{\underline{F}}^{-1}$$

Finally the variances of the parameters are used to obtain the corresponding "half-widths" w in Eq. 5, for instance for the parameter m:

$$w = \frac{\sqrt{Var(m)}}{m},$$

from which the confidence bounds can be calculated as described above: $m_{\mathrm{UB}}, m_{\mathrm{LB}} = m \cdot \exp(\pm w \cdot erf^{-1}(\mathrm{CL}))$, and similarly for F_0.

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